Comment on ''Phase ordering in chaotic map lattices with conserved dynamics''

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Angelini, Pellicoro, and Stramaglia [Phys. Rev. E **60**, R5021 (1999)] claim that the phase ordering of two-dimensional systems of sequentially updated chaotic maps with conserved ''order parameter'' does not belong, for large regions of parameter space, to the expected universality class. We show here that these results are due to a slow crossover and that a careful treatment of the data yields normal dynamical scaling. Moreover, we construct better models, i.e., synchronously updated coupled map lattices, which are exempt from these crossover effects, and allow for precise estimates of persistence exponents in this case.

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I. INTRODUCTION

The dynamical phenomenon of domain growth occurs in many different physical contexts. Once a fairly well-established subject [1], it has been recently the center of renewed interest for several distinct reasons: one is the advent of new quantifiers of the associated dynamical scaling regimes, such as first-passage or persistence exponents, i.e., the rate of algebraic decay of the probability for a given point in space to have remained in the same phase since some initial time [2]. Another reason is the natural question of the extent of known ''universality classes'' to new types of systems, e.g., spatiotemporally chaotic ones.

In this context, the recent study of the ordering properties of chaotic coupled map lattices (CMLs) possessing several symmetric phases in competition brought up some intriguing results [3]: in the simple case of two competing phases and a nonconserved order parameter, the ''normal'' growth law $L(t)\sim t^{1/z}$ with $z=2$, where $L(t)$ is the single length scale characterizing the coarsening pattern, was observed but with some exponent $z\geqslant 2$ continuously varying with parameters. However, this was later shown [4] to be only a (slow) transient behavior due to the nontrivial effect of space-discretization in these deterministic systems. For larger lattices and longer times than those considered in [3], L was shown to behave normally when plotted against $t^{1/2}$.

In a recent paper, Angelini, Pellicoro, and Stramaglia (APS) [5], motivated by the above study of nonconserved order parameter CMLs, presented a class of sequentially updated lattices of chaotic maps designed to investigate the case where the order parameter is locally conserved. In this case, L is also expected to grow algebraically with time, but with $z=3$ [1,6]. APS claim, however, that larger exponents are commonly found. Here we show that APS were misled by their treatment of data and that in fact the normal ($z=3$) growth law is observed in all cases. We argue, moreover, that fully deterministic, synchronously updated, coupled map lattices that conserve the order parameter can be easily constructed following the ideas of Oono and Puri [7], and we show that these systems behave very smoothly, enabling the precise measurement of persistence exponents in this context.

II. REVISITING THE APS RESULTS

A. The hybrid map lattices of APS

The lattices of maps introduced by APS are hybrid in several ways: a given local map f is first applied to all sites x_i of the lattice (a deterministic and synchronous operation), then pairs of nearest neighbors are sequentially and regularly visited and swapped probabilistically. (The regularity of the sweeps of the lattice is at the origin of the anisotropy of the domains in Fig. 1 of [5].) Furthermore, such systems are not ''coupled'' maps as in usual CMLs, since the values taken by the sites are not influenced by the swaps (they are always taken according to the invariant measure of the local map).

These systems are designed to mimick Ising systems (with fluctuating couplings) corresponding to the ''spins'' $\sigma_i=\mathrm{sgn}(x_i)$. The local map f has not much importance, and it is convenient to choose, following [8,3], an odd map of the $[-1,1]$ interval with two symmetric attractors. The energy of one configuration is given by $E=-\Sigma_{\langle i,j\rangle}x_i x_j$ where the sum is over nearest-neighbor pairs. The exchange probability reads $P_{\mathrm{swap}}=1/[1+\exp(\beta\Delta E)]$ where β is the inverse temperature and ΔE is the energy change of the swap. The zero-temperature limit is deterministic: swaps are effective if and only if they decrease the energy.

B. Domain growth is normal

We have performed numerical simulations of the APS system at zero temperature with the piecewise linear local map used in [3].

$$f(x)=\begin{cases}\mu X & \text{if} \quad X\in[-1/3,1/3]\\ 2\mu/3-\mu X & \text{if} \quad X\in[1/3,1]\\ -2\mu/3-\mu X & \text{if} \quad X\in[-1,-1/3],\end{cases} \tag{1}$$

with $\mu=1.9$. Coarsening occurs, with, again, a strong anisotropy due to the mode of update. The growth of L, defined as the width at midheight of the two-point autocorrelation function, is slow at short times, but then reaches the expected $t^{1/3}$ behavior [Fig. 1(a)], contrary to the claims of APS. The short-time behavior may be mistaken for anomalously slow algebraic growth (with an exponent close to the value $1/z=0.07$ reported by APS) when logarithmic scales are used, but a closer inspection shows a systematic increase of the

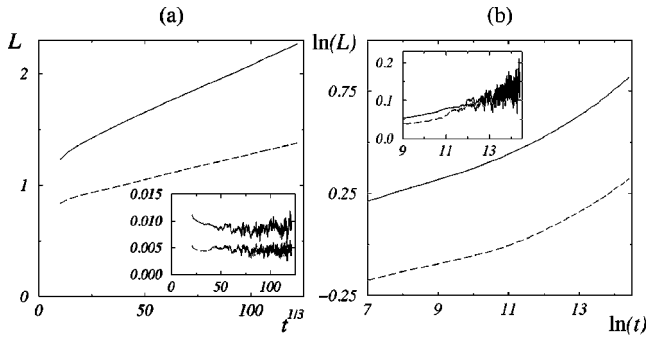

FIG. 1. Domain growth in the APS system with the local map (1) at zero temperature ($\beta=\infty$). Lattice of size 1024^2 with periodic boundary conditions. Solid lines: L defined as the width at mid-height of the two-point correlation function estimated along the x axis of the lattice. Dashed lines: same but along the y axis of the lattice. (a) L vs $t^{1/3}$; inset: local slopes. (b) $\ln(L)$ vs $\ln(t)$; inset: local exponent.

local exponent [Fig. 1(b)]. As a matter of fact, the system is so anisotropic that domains elongate in time [Fig. 1(a)]. This is due to our choice of updating x-wise pairs before y-wise pairs. Alternating this order would presumably suppress this effect.

Runs of the same system at finite temperatures indicate that domain growth is faster and that the crossover to the $z=3$ behavior occurs sooner. We are confident that similar results hold for the complicated local map mostly used by APS. As a conclusion, Fig. 3 of [5] has to be replaced by the variation of the *prefactor* of the $L\propto t^{1/3}$ law, similarly to the final conclusions of [4] for the nonconserved order-parameter case.

C. Persistence scaling is hard to measure

The persistence probability $p(t)=\mathrm{Prob}\{\sigma_i(t')=\sigma_i(t_0),\ \forall\ t'\in[t_0,t]\}$ is usually observed to decay algebraically with time [$p\sim(t/t_0)^{-\theta}$] in systems with algebraic growth laws. But persistence scaling for conserved order-parameter systems is notoriously difficult to observe [9]. An additional difficulty lies in the fact that the available models only show coarsening at finite temperatures, so that one has to resort to block-scaling of persistence. This is also the case of APS systems, *even at zero temperature*, since the chaotic fluctuations of the ''couplings'' amount to a finite temperature. This, by the way, is the reason why APS systems coarsen in this case.

Needless to say, the estimates of persistence exponents presented in [5] are then highly unreliable, if only because of the slow crossover for the growth law of L. Ideally, since persistence is a complex quantity involving all times since the reference time t_0, one should in principle choose t_0 in the asymptotic scaling regime and simulate the system up to times $t\gg t_0$. Given the typical values of the crossover times (Fig. 1), this is hardly possible. Another difficulty for the APS systems is the possible influence of their strong anisotropy on the persistence exponent θ [10].

Rather than trying to properly measure persistence scaling in APS systems, a possible but difficult task, we now turn ourselves to truly deterministic models, i.e., regular coupled map lattices, which are devoid of the drawbacks underlined above for APS systems.

III. WELL-BEHAVED DETERMINISTIC MODELS

A. Oono-Puri style CMLs

Deterministic models for phase ordering of conserved systems were introduced by Oono and Puri [7]. Dynamical scaling was observed to hold for these CMLs, with a growth law compatible with the expected $z=3$. We now present similar models that, in addition, can be constructed for any local map.

A usual CML, such as that studied in [3,4] for the non-conserved order-parameter case, can be written

$$x_i^{t+1}=\mathcal{F}(x_i^t)\equiv(1-\mathcal{N}g)f(x_i^t)+g\sum_{j\sim i}f(x_j^t), \tag{2}$$

where g is the coupling strength, $\mathcal{N}$ is the number of neighbors in the chosen coupling range, and the sum is over these neighbors.

Following [7], a CML conserving exactly the continuous local field can easily be constructed as

$$x_i^{t+1}=\mathcal{F}(x_i^t)-\frac{1}{\mathcal{N}}\sum_{j\sim i}[\mathcal{F}(x_j^t)-x_j^t], \tag{3}$$

where the last term corresponds to the extra Laplacian in the Cahn-Hilliard equation. Conservation of the continuous field x is obvious. On the other hand, strictly speaking, the discrete field $\sigma=\mathrm{sgn}(x)$ is not exactly conserved and fluctuates slightly, because the last term in Eq. (3) may change the sign of sites situated in domain walls.

The synchronous mode of update prevents excessive (i.e., other than lattice-derivated) anisotropy. The above structure ensures that ''true'' zero-temperature regimes are observed if the local map possesses two disjoint attractors. Changing the nature of these attractors (fixed points, limit cycles, chaotic sets), one can study competition between phases of different natures.

B. An example

We now present results obtained on a particular case of the models defined above. More comprehensive results will be reported elsewhere [11]. For simplicity reasons, we again choose the map given by Eq. (1). For ''extra smoothness,'' the Moore neighborhood on the square lattice ($\mathcal{N}=8$ neighbors of equal weight) was used.

As in [3,4], domain walls are strictly pinned for small g. For too strong coupling, on the other hand, antiferromagnetic-like phases appear. There is, however, an intermediate range of g values for which domain growth proceeds forever between two weakly chaotic phases. The expected $z=3$ law is then easily observed even at short times and in log-log scales [Fig. 2(a)].

The above CMLs reveal their strongest advantage when persistence scaling is considered. As already noticed, they show normal coarsening with true zero temperature. This allows us to avoid studying block persistence scaling, a somewhat tedious task. Figure 2(b) shows persistence decay for different reference times t_0. Nice scaling is easily observed. This constitutes, to our knowledge, the first clean evidence of algebraic decay of persistence in a two-dimensional conserved order-parameter system. Our results

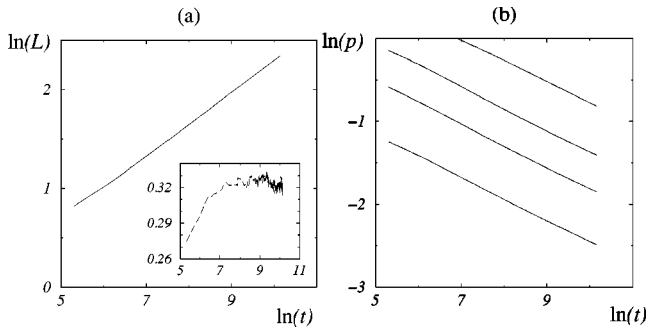

FIG. 2. Phase ordering and persistence scaling in a Oono-Puri style CML. Lattice of size 8192^2 with periodic boundary conditions. (a) $\ln(L)$ vs $\ln(t)$; inset: local exponent (The slightly smaller values of $1/z$ recorded can be shown to be due to space discretisation effects.) (b) $\ln(p)$ vs $\ln(t)$ for various initial times t_0. From bottom to top: $t_0=0$, 10, 10^2, and 10^3.

give an exponent $\theta \simeq 0.25(2)$, i.e., a value larger than that observed for the nonconserved order parameter case (for which $\theta \simeq 0.20-0.22$ [9,12]).

The above CMLs constitute an excellent base for a reliable study of persistence scaling in conserved order-parameter systems. Ongoing work is probing the degree of universality of the persistence exponent measured above, alongside a similar study for the nonconserved order-parameter case for which this issue is still unresolved [9,12].

IV. CONCLUSION

In this Comment, we showed how APS were misled in their interpretation of simulation data and that their conclusions about possible nontrivial values of the dynamical exponent z in chaotic systems with conserved order parameter dynamics do not hold. To a large extent, these systems can be seen as too close to the Ising model with Kawasaki dynamics, which is well known to be difficult to study numerically (although $z=3$ scaling is now well documented [13]).

We introduced a class of CMLs that are devoid of all the problems encountered in APS systems and that show normal scaling already at early times. Moreover, these systems also present nice scaling behavior for the persistence probability, whereas similar investigations in APS systems are riddled with problems. We showed unambiguously that two-dimensional systems with conserved order-parameter domain growth show an algebraic decay of persistence. Future work will try to assess the universality of both the so-called Fisher-Huse exponent and the persistence exponent θ in such systems.

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